

Vanishing largest Lyapunov exponent and Tsallis entropy

NIKOS KALOGEROPOULOS[§]

Weill Cornell Medical College in Qatar
Education City, P.O. Box 24144
Doha, Qatar

Abstract

We present a geometric, model-independent, argument that explains why the Tsallis entropy describes dynamical systems having vanishing largest Lyapunov exponent. We employ the Jacobi/geodesic deviation equation for an effective negative curvature Riemannian metric reflecting the Tsallis entropy composition property, whose solution gives the desired result. Extending the essential parts of the argument from Riemannian manifolds to CAT(k), $k < 0$ spaces, we see that the conclusion remains valid in the case of interacting systems described by different entropic parameters. This conclusion is in agreement with all currently known results.

PACS: 02.10.Hh, 05.45.Df, 64.60.al

Keywords: Tsallis entropy, Lyapunov exponents, Nonextensive statistical mechanics, CAT(k).

[§] E-mail: nik2011@qatar-med.cornell.edu

1. INTRODUCTION

The Tsallis entropy is a single-parameter family of functionals, first introduced in the Physics literature in 1988 [1], providing an alternative to the Boltzmann/Gibbs/Shannon (BGS) entropy used in the statistical description of a system. Consider a probability distribution $\{p_i\}$, $i \in I$ in a discrete sample space indexed by $I \subset \mathbb{N}$. Its Tsallis entropy is defined as

$$S_q = k_B \frac{1}{q-1} \left(1 - \sum_{i \in I} p_i^q \right) \quad (1)$$

It may be worth comparing (1) with the BGS entropy

$$S_{BGS}(\{p_i\}) = -k_B \sum_{i \in I} p_i \log p_i \quad (2)$$

where we immediately observe as the non-extensive/entropic parameter $q \rightarrow 1$, we get

$$\lim_{q \rightarrow 1} S_q = S_{BGS} \quad (3)$$

An analogous definition of the Tsallis entropy can be given for continuous sample spaces. Henceforth, we will be setting the Boltzmann constant $k_B = 1$, for simplicity.

The Tsallis entropy, conjecturally, describes collective phenomena with long-range spatial and temporal correlations [2], [3], systems whose phase portraits exhibit a fractal-like behavior etc. for which there is no reason or justification why their description by the BGS entropy should be accurate or even valid [3] (and references therein). Following the approach and viewpoint of Boltzmann [4], [5], one can state that the dynamical basis of the Tsallis entropy remains unclear so far [3]. This mirrors the existing difficulties in deriving the BGS entropy [3] - [6] from dynamical principles. As in the latter case, in the case of the Tsallis entropy, some progress has been made in identifying characteristics of systems effectively described by it. One class of such systems are ones exhibiting “weak chaos” or being at the “edge of chaos” for some subset of their parameter space [2], [3], [7] - [10]. This is more accurately expressed by referring to them as dynamical systems having vanishing largest Lyapunov exponent [3], [7] - [10].

A general, model-independent and formal justification of why such systems are described by the Tsallis entropy has been lacking so far [3], [11]. This is the issue that we are addressing in the present work. To be more specific, we present an argument about the converse: we argue, in Section 2, that if a system is described by the Tsallis

entropy, then its largest Lyapunov exponent vanishes. This statement is in agreement with all currently known results* about the underlying dynamics of systems described by the Tsallis entropy. In Section 3, we extend this conclusion to interacting systems having different values of q . Such interacting systems should be modelled on $\text{CAT}(k)$, $k < 0$ spaces, as we pointed out in [12]. Section 4, contains some general comments and points toward further implications of the Tsallis entropy composition property (6), employing the underlying concept of hyperbolicity.

For brevity, we do not provide the required background in Riemannian geometry or the metric geometry of $\text{CAT}(k)$, $k < 0$ spaces, referring instead to some of the excellent references such as [13], [14] and [15], [16] respectively, on these topics.

2. RIEMANNIAN SPACES: VANISHING LARGEST LYAPUNOV EXPONENT

One of the features distinguishing the Tsallis (1) from the BGS (2) entropy is their different composition properties. This distinction has profound consequences for the definitions of independence and additivity/extensivity that pervade Statistical Mechanics and Thermodynamics [3] (and references therein). Two systems A and B are conventionally defined to be “independent” if in their statistical description, the corresponding probability distributions p_A , p_B have as composition law the ordinary multiplication

$$p_{A+B} = p_A \cdot p_B \quad (4)$$

Here and henceforth $A + B$ indicates the compound system formed by combining A and B . For such independent systems (4), the BGS entropy is additive

$$S_{BGS}(A + B) = S_{BGS}(A) + S_{BGS}(B) \quad (5)$$

as can be immediately seen from (2). By contrast, under the same definition of independence (4), the Tsallis entropy (1) obeys the generalized additivity/composition property

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \quad (6)$$

This composition motivated the introduction of a generalized addition [17], [18] by

$$x \oplus_q y = x + y + (1 - q)xy \quad (7)$$

*We are grateful to Professor Constantino Tsallis for pointing this out to us.

What was lacking for sometime though, was an appropriate generalized counterpart to the usual multiplication that would be distributive with respect to (7). Despite some initial misses, two such generalized multiplications were introduced independently in [19], [20]. Although conjecturally independent, an explicit equivalence between them is still lacking. In these works a field isomorphism was, essentially, introduced that was indicated by τ_q in [20], where some of its rudimentary metric and measure-theoretical properties on \mathbb{R} were examined. In our subsequent work [12], we developed further some of the metric consequences of τ_q , $0 \leq q < 1$, initially for \mathbb{R}^n . Motivated by an analogy with the translation invariance of the Euclidean metric, we constructed in [12] a Riemannian metric \mathbf{g} , induced by (6), with components

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2tx} \end{pmatrix} \quad (8)$$

and corresponding line element

$$ds^2 = dx^2 + e^{-2tx} dy^2 \quad (9)$$

where

$$t = \log(2 - q) \quad (10)$$

The BGS composition property (“ordinary addition”) is expressed, in this formalism, via the effective Euclidean metric

$$\mathbf{g}_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (11)$$

whose corresponding line element is

$$ds_E^2 = dx^2 + dy^2 \quad (12)$$

The two metrics (8), (11) were used in [12] to encode and compare, in metric terms, the composition properties of the BGS and the Tsallis entropies.

The metric tensor (8) turned out [12] to have a constant negative sectional curvature

$$k = -[\log(2 - q)]^2 \quad (13)$$

which provided a geometric interpretation of the non-extensive parameter $q \in [0, 1)$. Endowing the plane \mathbb{R}^2 with (8), turned it into a re-scaled version of the hyperbolic plane \mathbb{H}^2 . Then, we concluded in [12], that due to (6) which gave rise to (8), the Tsallis

entropy can be thought as a “hyperbolic analogue” of the BGS entropy.

Without adding any complexity to the subsequent argument, we will consider in the formalism instead of just \mathbb{R}^2 , a general Riemannian manifold (M, \mathbf{g}) with tangent bundle indicated by TM . Let the Levi-Civita connection compatible with \mathbf{g} be indicated by ∇ . Such a connection is expressed in terms of \mathbf{g} by the Koszul formula

$$\begin{aligned} 2\mathbf{g}(\nabla_X Y, Z) = & -\mathbf{g}(X, [Y, Z]) + \mathbf{g}(Z, [X, Y]) + \mathbf{g}(Y, [Z, X]) \\ & + X[\mathbf{g}(Y, Z)] - Z[\mathbf{g}(X, Y)] + Y[\mathbf{g}(X, Z)] \end{aligned} \quad (14)$$

where $X, Y, Z \in TM$. The geodesic equation is [13], [14]

$$\nabla_X X = 0 \quad (15)$$

for $X \in TM$ tangent to the geodesic. If $J \in TM$ indicates a Jacobi field then it satisfies the Jacobi/geodesic deviation equation [13], [14]

$$\nabla_X \nabla_X J + R(J, X)X = 0 \quad (16)$$

Here $R(X, Y)Z$ indicates the Riemann tensor which is defined by [13], [14]

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z \quad (17)$$

It is extremely difficult to explicitly solve either the geodesic (15) or the Jacobi equations (16), except in a few particularly simple cases, our case of interest being one of them. Let e_1, e_2 be orthonormal vectors, with respect to \mathbf{g} , spanning a 2-dimensional subspace of TM in a neighborhood of $x \in M$. The sectional curvature of M in this 2-plane subspace of TM is defined by [13], [14]

$$k = \mathbf{g}(R(e_1, e_2)e_2, e_1) \quad (18)$$

In our case of interest, initially at least, $k < 0$ is constant. Then the Jacobi equation (16) reduces to the ordinary differential equation

$$\frac{d^2 J(s)}{ds^2} - kJ(s) = 0 \quad (19)$$

which has the general solution

$$J(s) = \sum_{i=1}^{n-1} \left\{ a_i \exp(\sqrt{-k} s) + b_i \exp(-\sqrt{-k} s) \right\} e_i(s) \quad (20)$$

where a_i, b_i are constants, $\{e_i(s)\}$ are parallel orthonormal vectors (Fermi basis) and s is the arc-length parameter of the geodesic whose tangent is $X(s)$. The summation takes place over the $n - 1$ directions orthogonal to $X(s)$. Substituting (10) into (20), we find that

$$J(s) = \sum_{i=1}^{n-1} \{a_i(2 - q)e^s - b_i(2 - q)e^s\} e_i(s) \quad (21)$$

Hence, we see that in a Riemannian manifold of constant negative curvature (13), the nearby geodesics deviate from each other exponentially in terms of the geodesic arc-length s or any of its affine re-parametrizations. The Riemannian metric (8) is a special case of the general \mathbf{g} , which is defined on $M = \mathbb{R}^2$. This exponential deviation of the nearby geodesics of a manifold of negative sectional curvature should be contrasted to those of the Euclidean metric (11) which has $k = 0$: in the Euclidean case the geodesics separate linearly as functions of the arc-length parameter s , as can be immediately seen from (19) by setting $k = 0$.

Let's rephrase the above argument in an alternative, coordinate dependent, way. Consider a unit vector with respect to (11). This vector will have a smaller magnitude with respect to (8), as can be immediately seen. So (8) does not increase the magnitudes of the unit vectors. Instead, it exponentially decreases their y -component. As a result, it will increase distances of any rectifiable curve in the y /transversal direction to x , by an exponential factor, which is exactly the statement contained in (20).

We turn our attention to Lyapunov exponents. We will only be needing their definition for the case of uniformly hyperbolic dynamical systems [21], [22], a standard example of which is the geodesic flow on a Riemannian manifold of, generally variable, negative sectional curvature. To be slightly more general than that, let $f_t : M \rightarrow M$ be a flow on the Riemannian manifold (M, \mathbf{g}) , whose generating vector field is

$$X(t) = \frac{d}{dt}(f_t(x))|_{t=0} \quad (22)$$

Let $Y \in TM$ and let its norm with respect to the Riemannian metric \mathbf{g} be indicated by $\|Y\| = \{\mathbf{g}(Y, Y)\}^{\frac{1}{2}}$. The Lyapunov exponent of the perturbation in the direction of $Y \in T_x M$ along the evolution/trajectory of the flow f_t is defined by

$$\lambda_x(Y) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|d_x(f_t Y)\| \quad (23)$$

The Lyapunov exponent measures the asymptotic rate of change of the magnitude of a perturbation in the direction of Y . It is clear that if someone is interested in a stability analysis of a flow, the most pertinent Lyapunov exponent is the largest positive one in some transversal direction. Consider, as a special case, f_t to be the geodesic flow on \mathbb{R}^2 endowed with (8) that was discussed above. We see from (21) that

$$\lim_{s \rightarrow \infty} \|J(s)\| = e^s \quad (24)$$

therefore, the definition (23) gives

$$\lambda = 1 \quad (25)$$

To summarize: we started from a dynamical system modelled on a Riemannian manifold M equipped with a metric induced by (11). We assumed that this dynamical system's effective statistical description is provided by the Tsallis entropy (1). In turn, the Tsallis entropy composition property (6) indicated that it is more suitable to use the effective hyperbolic metric (8) on M manifold, instead of the original one (11). It was the collective description of the underlying dynamics that dictated this “hyperbolization” in producing the effective metric (8) from (11). This “hyperbolization” was concretely implemented as the “warping” by a convex function, the exponential in the present case, in transitioning from (11) to (8). The instabilities of the dynamical system were expressed through its positive Lyapunov exponents, initially with respect to (11), the largest of them being the most important. When the effective behavior described by the hyperbolic metric (8) is taken into account, the largest positive Lyapunov exponent of the underlying dynamical system becomes zero, as can be seen in (24) and (25). The reason is that the perturbations of the underlying dynamical system and the distances with respect to the hyperbolic metric (8) grow at the same rate (25). Hence their relative growth rate is linear, as seen in (25), so the Lyapunov exponent of the underlying dynamical system relative to (8) is zero. This is the main conclusion of the present work and it is in agreement with all known results reached by analyzing particular models [3] (and references therein).

In case the largest Lyapunov exponent is zero, as in the case discussed above, the instabilities of the dynamical system grow at a milder than exponential rate, with respect to the hyperbolic metric (8), and consequently their asymptotic behavior has to be encoded differently, if we want to obtain non-trivial results. One way to quantify the growth of such perturbations by modifying the definition of Lyapunov exponents (23) to

$$\tilde{\lambda}_x(Y) = \lim_{t \rightarrow \infty} \frac{\log \|d_x f_t Y\|}{\log t} \quad (26)$$

These modified exponents describe perturbations obeying an asymptotic, power-law evolution $t^{\tilde{\lambda}_x(Y)}$. This is essentially the definition adopted in [3], [7] - [9]

$$\frac{d\xi}{dt} = \lambda_q \xi^q \quad (27)$$

for systems described by the Tsallis entropy, in just different notation. Naturally, the modified Lyapunov exponents (26) will explicitly depend on the value of the entropic parameter q of the dynamical system. Since, as is conjectured, the systems described by the Tsallis entropy may possess more than just one value of non-extensive parameter q [3], [11], depending on which property of the system is described by them, the modified Lyapunov exponents (26) would evidently depend on the value of q determining the sensitivity of the system to infinitesimal perturbations, indicated as q_{sen} in [3].

The use of a definition like (26) would be of limited interest for applications, if explicit constructions/examples did not exist for which it could provide non-trivial information. Motivated by the linear (exponential) increase of the geodesic distances in Riemannian manifolds of zero (negative resp.) sectional curvature (21), one may be wondering whether an intermediate behavior described by (26) is even possible. The answer turns out to be affirmative. An example of a quadratic separation of geodesics in a 2-complex endowed with a CAT(0) metric was constructed[†] in [23]. This quadratic geodesic deviation in a CAT(0) space is quite different from the case of CAT(k), $k < 0$, whose geodesics deviate exponentially from each other, as will be explained in the next Section. Whether the property of quadratic or, more generally, polynomial divergence of geodesics encoded in (26) is “typical”, or even common, for CAT(0) spaces, and whether in such a case the definition (26) exhausts all possible geodesic deviation behavior, does not seem to be known at this time. More importantly, it is unclear, to us at least, exactly what, if any, physical system the CAT(0) space construction of [23] can be used to describe. For these reasons, we refer to [23] for the construction itself and its mathematical aspects, aiming to re-visit this topic, if physical reasons warrant it in the future.

3. GEODESIC DEVIATION IN CAT(k), $k < 0$ SPACES

We continue with the generalization of the above results to the case of CAT(k), $k < 0$ spaces. The need for using CAT(k), $k < 0$ spaces, motivated by the composition of the

[†]We are grateful to Professor Panos Papasoglou for bringing this work to our attention.

Tsallis entropy (6), was explained in [12]. The major obstacle in repeating the Riemannian approach verbatim, is that $\text{CAT}(k)$ spaces do not possess a differential structure [15], [16], [24], so one has to dispense with statements relying on regularity properties, such as ones formulated via vector fields, the geodesic (15) and the Jacobi (16) equations etc. The only option left is to employ the triangle inequality, which when combined with the $\text{CAT}(k)$, $k < 0$ condition proves to be sufficient for attaining the sought-after goal [15], [16], [24]. One should notice, that due to lack of smoothness, most statements in the present Section can only be formulated via inequalities, as contrasted to the equalities, such as (20), derived in the Riemannian case. As a result, the arguments in this Section, applied to the case of $\text{CAT}(k)$, $k < 0$ spaces are, inevitably, synthetic as opposed to the analytic ones in the Riemannian case of the previous section. We will, largely, follow [24] in the sequel.

One begins by realizing that a geodesic space (\mathfrak{X}, d) which is $\text{CAT}(k)$, $k < 0$ is necessarily hyperbolic. There are several definitions of hyperbolicity at various levels of generality [24] and several equivalences among them [24]. The following definition, ascribed to E. Rips, is the most useful for our purposes: a metric space (\mathfrak{X}, d) is δ -hyperbolic, for $\delta > 0$, if for any three points $x, y, z \in \mathfrak{X}$, any side of the triangle having x, y, z as vertices lies in a δ -neighborhood of the union of the two others. Given this definition, the hyperbolicity of the $\text{CAT}(k)$, $k < 0$ space (\mathfrak{X}, d) follows immediately from the definition of the $\text{CAT}(k)$ condition.

Consider a polygon in such (\mathfrak{X}, d) with $n \in \mathbb{N}$ vertices x_1, x_2, \dots, x_n . Let

$$m = \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad (28)$$

where the square brackets indicate the integer part of their argument. Because (\mathfrak{X}, d) is a geodesic space, every segment has a midpoint. Consider the midpoint x_m and the triangle $x_1 x_m x_n$. Since (\mathfrak{X}, d) is δ -hyperbolic, there is a point y in the union of the two other sides $x_1 x_m \cup x_m x_n$ such that $d(y, z) \leq \delta$ where $z \in x_n x_1$. Without loss of generality, we assume that $y \in x_1 x_m$. Using induction on the m -polygon x_1, x_2, \dots, x_m , we find that the distance of z from the union of all other sides of this m -polygon is

$$d \left(z, \bigcup_{i=1}^{m-1} x_i x_{i+1} \right) \leq (p-1)\delta \quad (29)$$

where $p \in \mathbb{N}$ satisfies

$$p \geq \frac{\log(n-1)}{\log 2} \quad (30)$$

We conclude then, that in (\mathfrak{X}, d) each side of the n -polygon is contained in a $p\delta$ -neighborhood of the union of its other sides.

The second, and last, step is the quantification of the concept of the exponential separation of geodesics in (\mathfrak{X}, d) . Consider two segments emanating from $x \in \mathfrak{X}$ toward $y, z \in \mathfrak{X}$, respectively. Let two objects move with unit speed, one in each of these two arc-length parametrized segments. We are interested in the separation of x and y after time t . An obvious way to measure such a separation would be to start at the location of y , move back along the segment joining it with x by a distance t , switch segment at x and then continue from x alongside the other segment for an additional distance t until reaching z . This is not however, the analogue of a curve joining y and z of Section 2. What we want is to measure the “direct” separation between y and z , without having to go back close to the intersection x of the two segments. So, we want to determine the length of a path from y to z , in the complement of a ball $B_r(x)$ of radius $r > 0$ centered at x . Consider a path $x_1 x_2 \dots x_n$, $n \in \mathbb{N}$, such that $d(x_i, x_{i+1}) \leq \epsilon$, $i = 1, \dots, n-1$ which lies outside the ball $B_r(x)$, with $x \in x_n x_1$. Let $p \in \mathbb{N}$ be as in (30). Using the conclusion of the previous paragraph, there exists a point $w \in x_i x_{i+1}$, $i = 1, \dots, n-1$ such that $d(x, w) \leq p\delta$. Since

$$d(x, w) \geq r - \frac{\epsilon}{2} \quad (31)$$

we find

$$p \geq \frac{r}{\delta} - \frac{\epsilon}{2\delta} \quad (32)$$

which gives

$$n \geq 2^{p-1} \geq c \cdot 2^{\frac{r}{\delta}} \quad (33)$$

where

$$c = 2^{-(\frac{\epsilon}{2\delta} + 1)} \quad (34)$$

The conclusion that we reach from (33), is that the number of points with a uniform distance upper bound ϵ between two consecutive ones, making up the path between y and z outside $B_r(x)$ increases exponentially with r in the $\text{CAT}(k)$, $k < 0$ space (\mathfrak{X}, d) . This is what we wanted to show. We see that (33) is the analogue of (21) for (\mathfrak{X}, d) . Naturally, the argument leading to (33) is already applicable to the case of a

Riemannian manifold (M, \mathbf{g}) of negative sectional curvature, since in this case (M, \mathbf{g}) is a $\text{CAT}(k)$, $k < 0$ space. So, the conclusions drawn in the Riemannian case, about the effective metric behavior of systems described by the Tsallis entropy, can be extended unaltered to the case of any number of interacting systems described by different values of q .

4. CONCLUSIONS AND OUTLOOK

In the present work, we attempted to justify why dynamical systems whose statistical behavior is described by the Tsallis entropy, have vanishing largest Lyapunov exponent. This was essentially ascribed to employing the effective negative curvature metric (8), which is the “hyperbolization” of the Euclidean initially employed metric (11), as was pointed out in [12]. Our conclusion is in agreement with all currently known results.

The process of generalizing this conclusion to the case of $\text{CAT}(k)$, $k < 0$ spaces presented in Section 3, is of interest, as it points out to the underlying reason behind such behavior. The argument of Section 3 shows that the key in understanding consequences of the Tsallis entropy composition property (6) is the concept of hyperbolicity, which was also alluded to in [12]. Knowing this, it should not come as a surprise that the argument of Section 3 is a small part of a well-known proof of Morse’s Lemma, which establishes the stability of geodesics in hyperbolic geodesic spaces under quasi-isometries [15], [16], [24]. In particular, since Riemannian manifolds of negative sectional curvature (M, \mathbf{g}) are a subset of $\text{CAT}(k)$, $k < 0$ spaces, the vanishing of the highest Lyapunov exponent of the dynamical systems modelled by (M, \mathbf{g}) just expresses their underlying hyperbolicity in a very compact way. The general framework of this hyperbolicity and its implications for systems described by the Tsallis entropy will be examined further in a future work.

ACKNOWLEDGEMENT

The author is grateful to Professor Constantino Tsallis for emphasizing that it is the largest Lyapunov exponent that vanishes in systems described by the Tsallis entropy, but not necessarily all of them.

REFERENCES

- [1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)
- [2] A.M. Mariz, C. Tsallis, *Long memory constitutes a unified mesoscopic mechanism consistent with nonextensive statistical mechanics*, arXiv:1106.3100
- [3] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*, Springer (2009)
- [4] L. Boltzmann, Acad. Wissen. Wien, Math.-Naturwissen. **75**, 67 (1877)
- [5] G. Gallavotti, *Statistical Mechanics: A Short Treatise*, Springer (1999)
- [6] E.G.D. Cohen, Pramana **64**, 635 (2005)
- [7] P. Grassberger, M. Scheunert, J. Stat. Phys. **26**, 697 (1981)
- [8] G. Anania, A. Politi, Europhys. Lett. **7**, 119 (1988)
- [9] H. Hata, T. Horita, H. Mori, Prog. Theor. Phys. **82**, 897 (1989)
- [10] M.A. Fuentes, Y. Sato, C. Tsallis, *Sensitivity, entropy and escape rates at the onset of chaos*, Phys. Lett. A (in press), arXiv:1106.3761
- [11] C. Tsallis, *Some Open Points In Nonextensive Statistical Mechanics*, arXiv:1102.2408
- [12] N. Kalogeropoulos, Physica A **391**, 3435 (2012)
- [13] J. Cheeger, D.G. Ebin, *Comparison Theorems in Riemannian Geometry*, AMS Chelsea (1975)
- [14] T. Sakai, *Riemannian Geometry*, Amer. Math. Soc. (1996)
- [15] M.R. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer (1999)
- [16] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser (1999)
- [17] L. Nivanen, A. Le Mehauté, Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003)
- [18] E.P. Borges, Physica A **340**, 95 (2004)
- [19] T.C. Petit Lobão, P.G.S. Cardoso, S.T.R. Pinho, E.P. Borges, Braz. J. Phys. **39**, 402 (2009)
- [20] N. Kalogeropoulos, Physica A **391**, 1120 (2012)
- [21] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press (1995)
- [22] L. Barreira, Y. Pesin, *Dynamics of Systems with Nonzero Lyapunov Exponents*, Cambridge Univ. Press (2007)
- [23] S.M. Gersten, Geom. Funct. Anal. **4**, 37 (1994)
- [24] E. Ghys, P. de la Harpe, (Eds). *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhäuser (1990)